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SOME MIXED PROBLEMS OF THE THEORY OF THE BENDING OF PLATES ON AN ELASTIC FOUNDATION*

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Problems of the impression of one or two inclusions in the form of stiffener ribs into an infinite plate lying on an elastic foundation are studied. It follows from the properties of the kernels of the integral equations of these problems that their solutions have a non-integrable singularity /1/. By extending the method of "large λ" /2/, an asymptotic solution is first constructed for the integral equation in two sections. The numerical analysis performed can be useful when designing the structural elements of airfield road and hydrotechnical structures as well as buildings on the surface of an ice cover.

1. Let a thin stiff inclusion, settling by an amount $f(x)$ be imbedded by a force P along a segment $y=0$, $|x|\leqslant a$ in an infinite $(-\infty<x,\ y<\infty)$ Kirchhoff-Love plate lying on a Winkler foundation. When interaction occurs between the inclusion and the plate, contact forces $\varphi(x)$ occur that cause a break in the continuity of the generalized transverse forces. An integral equation in $\varphi(x)$ is obtained** (**Ismail, Kh.T., Boundary-value problems of the bending of plates on an elastic foundation in the presence of rectilinear defects. Candidate Dissertation. Odessa State University, 1986.) by applying a generalized Fourier integral transformation to the bending equation of plates on a Winkler foundation and the boundary conditions, and after inserting dimensionless quantities it has the form (primes are omitted)

$$\int_{-1}^{1}\varphi(y)\,k\left(\frac{y-x}{\lambda}\right)dy=\pi f(x)\quad(|x|\leqslant 1) \tag{1.1}$$

$$k(t)=\int_0^\infty K(u)\cos ut\,du,\quad K(u)=\frac{\sqrt{2}}{\sqrt{u^4+1}\sqrt{u^2+\sqrt{u^4+1}}},\quad \lambda=\frac{1}{a}\left(\frac{d}{s}\right)^{1/4}$$

$$(x=ax',\ y=ay',\ 2f(x)=af'(x'),\ \varphi(y)=\sqrt{ds}\varphi'(y'))$$

where d and s are the plate and foundation stiffnesses, respectively. This equation was reduced** (**Ismail, Kh.T., Boundary-value problems of the bending of plates on an elastic foundation in the presence of rectilinear defects. Candidate Dissertation, Odessa State University, 1986.) to an infinite system of linear algebraic equations and its solvability was proved. Following /2/, we here obtain explicit asymptotic formulas for $\varphi(x)$ in the case when $f(x)\equiv 1$. For $\lambda\geqslant 2$ we assume

$$k(t)={}^1/_2t^2\ln|t|-F(t) \tag{1.2}$$

$$F(t)=\frac{3}{4}t^2-\int_0^\infty u^{-3}\left((u^3K(u)-1)\cos ut+1-\frac{1}{2}u^2t^2e^{-u}\right)du$$

The expansion

$$F(t)=\sum_{i=0}^\infty a_it^{2i}+\ln|t|\sum_{i=2}^\infty d_it^{2i};\quad a_0=-1.514;\quad a_1=0.575$$

$$a_2=0.0242,\quad a_3=-0.00213,\quad d_2=0,\quad d_3=0.000868$$

can be obtained.

We will finally have

$$\varphi(x)=\lambda^2\frac{A_1x^2+A_2}{(1-x^2)^{5/2}}+\frac{1}{\sqrt{1-x^2}}\left\{24a_2A_1\left(x^2-\frac{1}{2}\right)+\frac{1}{\lambda^2}\left(3A_2\left(\frac{1}{2}-x^2\right)+\right.\right. \tag{1.3}$$

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$$A_1\left(\frac{19}{8}-4x^2-x^4\right)\Big)(120d_3\ln\lambda-74\,d_3-120a_3)+$$
$$\frac{120d_3}{\lambda^2}\Big(A_1\left(\frac{19}{8}\ln 2-\frac{143}{48}\right)+A_2\left(\frac{3}{2}\ln 2-\frac{17}{8}\right)+x^2\Big(A_1\left(\frac{23}{6}-4\ln 2\right)+$$
$$A_2\left(\frac{7}{2}-3\ln 2\right)\Big)-x^4\Big(A_1\left(\ln 2-\frac{17}{6}\right)-A_2\Big)\Big)\Big\}+O(\lambda^{-4}\ln^2\lambda)$$

The constants A_1, A_2 are determined from a system of two linear equations /2/. For instance, for $\lambda=2$ we have $A_1=-0.182$ and $A_2=-0.0103$.

For $0<\lambda<2$ we approximate $K(u)$ by expression (6.4) from /2/, where $A=0.703$, $B=1.48$, $C=0.625$, $D=0.649$, $E=2.26$, and $F=0.354$. The error of this approximation on the real axis is $\theta=11.5\%$, i.e., relatively high, which is caused by the disappearance of the coefficient of u^{-5} in the asymptotic expansion of $K(u)$ as $u\to\infty$. We hence find

$$\varphi(x)=\psi\left(\frac{1+x}{\lambda}\right)+\psi\left(\frac{1-x}{\lambda}\right)-\frac{0{,}396}{\lambda^2} \tag{1.4}$$
$$\psi(x)=\frac{0{,}629}{\lambda}\left(-\frac{e^{-Ax}}{2\sqrt{\pi x^3}}+0{,}14\frac{e^{-Ax}}{\sqrt{\pi x}}-0{,}477\chi_E(x)-0{,}133\chi_F(x)+\right.$$
$$\left.0{,}750\chi_0(x)\right),\quad \chi_G(x)=\sqrt{A-G}\,e^{-Gx}\operatorname{erf}\sqrt{(A-G)x},\quad G=E,F,0$$

The errors in the solutions (1.3) and (1.4) do not exceed 5 and $(5+\theta)\%$, respectively. As is seen from the values of $\varphi(x)$ presented below, juncture of the solutions (1.3) and (1.4) occurs for $\lambda=2$

x	0	0.1	0.3	0.5	0.7	0.9
$\varphi_{(1.3)}$	0.0106	0.0191	0.0968	0.326	1.09	7.66
$\varphi_{(1.4)}$	0.0122	0.0226	0.106	0.345	1.12	7.40

The plate deflections referred to a are given by the formula

$$\omega(x,y)=\int_0^\infty\int_{-1}^{1}\varphi(\alpha)\cos\frac{u\alpha}{\lambda}\,d\alpha\cos\frac{ux}{\lambda}\exp\left(\frac{-\sigma_+|y|}{\lambda}\right)\times \tag{1.5}$$
$$\left(\sigma_+\sin\frac{\sigma_-|y|}{\lambda}+\sigma_-\cos\frac{\sigma_-y}{\lambda}\right)\frac{du}{\sqrt{u^4+1}},\quad \sigma_\pm=\frac{\pm u^2+\sqrt{u^4+1}}{2}$$

The inner integral in (1.5) is understood in the sense of the finite part /2, 3/. The bending moments M_x and M_y referred to d/a are easily calculated for a known function $\omega(x,y)$. By virtue of the symmetry of the problem, the quantities $|M_x|$, $|M_y|$ have a local maximum at the origin. Moreover, as computations performed for different $\lambda>0$ show, on any ray emerging from the origin, $|M_x|$, $|M_y|$ have an infinite number of local maxima that damp out at infinity. As a rule, the absolute maximum is the first of second local maximum. Diagrams of equal maximum moments can be constructed that are similar in shape to the arcs of an ellipse. The plate sections along these contours are most dangerous for the design of structures. An analogous effect is known in problems on the action of a concentrated force on a beam or slab lying on an elastic foundation /4, 5/. It is here necessary to set $\varphi(x)$ equal to the Dirac δ- function in (1.5); the bending moments will be infinite at the point of concentrated force application.

Certain values of M_x are given below for $\lambda=5$ and Poisson's ratio $\nu=0.3$. It is seen that the local maximum is reached on the line $x=y$ for $x=0.76$ while it is at $y=1.36$ on the ordinate axis:

x	0	0.5	0.76	1.0	1.2	2.0
$M_x(x,\ x)$	−1.162	−0.0971	−0.0211	−0.0259	−0.0214	0.00281
y	0	0.5	1.0	1.36	1.5	2.0
$M_x(0,\ y)$	−1.162	−0.151	0.0806	0.101	0.100	0.0830

We also note that the bending moments along a rib are considerably greater than on a line perpendicular to it.

2. The integral equation of the problem of impressing two ribs along the segments $y=0$, $a\leqslant|x|\leqslant b$ in an infinite plate lying on a Winkler foundation can obviously be separated

into even and odd cases

$$\int_k^1 \varphi_\pm(y)\left[k\left(\frac{y-x}{\lambda}\right)\pm k\left(\frac{y+x}{\lambda}\right)\right]dy=\pi f_\pm(x)\quad (k\leqslant x\leqslant 1) \tag{2.1}$$
$$2\varphi_\pm(x)=\varphi(x)\pm\varphi(-x),\quad 2f_\pm(x)=f(x)\pm f(-x)$$
$$k=a/b,\quad \lambda=(d/s)^{1/4}/b$$

To solve (2.1) by the method of "large λ" we separate the ideas from /2/. Taking account of the properties of the function $k(t)$ /2/, we rewrite (2.1) in the form

$$\int_k^1 \varphi_\pm^*(y)\left[\frac{(y-x)^2}{2}\ln\frac{|y-x|}{\lambda}\pm\frac{(y+x)^2}{2}\ln\frac{(y+x)}{\lambda}\right]dy=\pi G_\pm(x) \tag{2.2}$$
$$(k\leqslant x\leqslant 1)$$
$$\lambda^2\varphi_\pm^*(x)=\varphi_\pm(x)$$
$$G_\pm(x)=f_\pm(x)+\lambda^2\int_k^1\varphi_\pm^*(y)\left[F\left(\frac{y-x}{\lambda}\right)\pm F\left(\frac{y+x}{\lambda}\right)\right]dy$$

We introduce the functions (the upper and lower term in the braces is taken for the superscript, or subscript, respectively)

$$\psi_\pm(x)=\varphi_\pm^*(x)-\frac{\operatorname{sgn}x\,(\alpha_\pm+\beta_\pm x^2)}{g^3(x)}\begin{Bmatrix}x\\1\end{Bmatrix} \tag{2.3}$$
$$g(x)=\sqrt{(1-x^2)(x^2-k^2)}$$

where $\alpha_\pm$ and $\beta_\pm$ are defined such that the functions $\psi_\pm(x)$ have integrable singularities at the points $\pm k$, ± 1.

We substitute (2.3) into (2.2) and, differentiating the equation obtained three times with respect to x, we arrive at equations of a well-known type /6/ in $\psi_\pm(x)$, which we solve to obtain inversion formulas for the integral operator (2.2)

$$\varphi_\pm^*(x)=\left(\frac{2(Q_\pm+R_\pm x^2+T_\pm x^4)}{g^3(x)}+\right. \tag{2.4}$$
$$\left.\frac{2}{\pi g(x)}\int_k^1\frac{g(y)\,G_\pm'''(y)}{y^2-x^2}\begin{Bmatrix}1\\y\end{Bmatrix}dy\right)\begin{Bmatrix}|x|\\\operatorname{sgn}x\end{Bmatrix}\quad(k\leqslant x\leqslant 1)$$

The constants $Q_\pm, R_\pm, T_\pm$ are found from systems of three linear equations (omitted here because of their length), which must be derived by acting on (2.2) with the following operators:

$$\begin{Bmatrix}x\\1\end{Bmatrix}\frac{dx}{g(x)}\int_k^1\cdot,\quad \begin{Bmatrix}x\\1\end{Bmatrix}\frac{x^n\,dx}{g^3(x)}\int_k^1\cdot\qquad (n=0,2)$$

The need to evaluate the following integrals therefore arises:

$$b_n=\int_k^1 t^{2n}\frac{dt}{g^3(t)}, \tag{2.5}$$
$$c_n=\frac{\operatorname{sgn}y}{\pi}\int_k^1 t^{2n}\ln\left|\frac{t-y}{t+y}\right|\frac{dt}{g^3(t)}\qquad(k<|y|<1,\ n=0,1\ldots)$$

The integrals (2.5) are understood in the sense of the finite part. For $n\geqslant 2$ they are expressed in terms of b_0, b_1, c_0, c_1 and elliptic integrals. Their values for $n=0,1$, computed on a computer, are presented below

k	0.1	0.3	0.5	0.7	0.9
$-b_0$	97.15	9.689	3.097	1.409	0.7676
b_1	1.735	0.8106	0.4862	0.3233	0.2271
$-c_0$	2.402	2.814	4.055	8.467	57.73
$-c_1$	0.9233	0.9307	1.115	1.628	0.5192

Expanding $\varphi_\pm^*(x)$ in the form

$$\varphi_\pm^*(x) = \sum_{i=0}^{2} \sum_{j=0}^{[i/2]} \varphi_{2i,j}^{\pm}(x) \lambda^{-2i} \ln^j\lambda + O(\lambda^{-4}\ln^2\lambda) \tag{2.6}$$

substituting (2.6) into (2.4) and equating terms of identical powers of $\lambda \ln \lambda$ we successively determine the function-coefficients in the expansion (2.6).

For instance, for $f_+(x) \equiv 1$ we find that

$$\varphi_{00}^+(x) = \frac{2|x|}{g^3(x)} (Q_+ + R_+x^2 + T_+x^4), \quad \varphi_{20}^+(x) = \frac{48a_2|x|}{g(x)} T_+h_2(x) \tag{2.7}$$

$$\varphi_{21}^+(x) \equiv \varphi_{42}^+(x) \equiv 0$$

$$\varphi_{41}^+(x) = -\frac{240d_3|x|}{g(x)} \left(T_+h_4(x) + 3\left(R_+ + T_+\frac{3}{2}(1+k^2)\right)h_2(x)\right)$$

$$\varphi_{40}^+(x) = \frac{|x|}{g(x)} \left((240a_3 + 228d_3)T_+ + 120d_3\left(\frac{4Q_+}{(1-k^2)^2} + \frac{2(1+k^2)}{(1-k^2)^2}R_+ + f_0T_+\right)\right)h_4(x) +$$

$$\frac{|x|}{g(x)}\left((720a_3 + 324d_3)\left(R_+ + T_+\frac{3}{2}(1+k^2)\right)\right)h_2(x) + 240d_3\left(R_+ + \frac{3}{2}T_+\right) +$$

$$120d_3\left(\frac{2(1+k^2)}{(1-k^2)^2}Q_+ + f_0R_+ - f_1T_+\right)\Bigg) - \frac{480d_3|x|}{\pi g(x)} \int_k^1 \frac{g(y)\,dy}{y^2 - x^2}(f_2Q_+ + f_3R_+ + f_4T_+) -$$

$$\frac{1440d_3|x|}{\pi g(x)} \int_k^1 \frac{y^2g(y)\,dy}{y^2 - x^2}(c_1Q_+ + f_2R_+ + f_3T_+)$$

$$h_2(x) = x^2 - \frac{1+k^2}{2}, \quad h_4(x) = x^4 - x^2\frac{1+k^2}{2} + \frac{(1-k^2)^2}{8}$$

$$f_0 = \ln\frac{1-k^2}{4} + \frac{2(1+k^2)}{(1-k^2)^2}, \quad f_1 = \frac{(1+k^2)(6k^2 - 5k^4 - 5)}{2(1-k^2)^2} - \frac{3}{2}(1+k^2)\ln\frac{1-k^2}{4}$$

$$f_2 = K - k^2c_0 + (1+k^2)c_1, \; f_3 = K - E - k^2c_1 + (1+k^2)f_2, \; f_4 = K\left(\frac{8}{3} + \frac{7k^2}{3} + \frac{3k^4}{4}\right) -$$

$$\frac{5}{3}E(1+k^2) - c_0k^2(1+k^2+k^4) + c_1(1+k^2)(1+k^4)$$

Here $K = K(k)$ and $E = E(k)$ are complete elliptic integrals. The singular integrals in the formula for $\varphi_{40}^*(x)$ /7/ can also be expressed in terms of elliptic integrals.

The solution (2.6) yields an error not greater than 5% for $\lambda \geqslant 2$.

Starting from (2.4), we will derive asymptotic formulas for the forces $P_\pm$ and the moments $M_\pm$ acting on the inclusions. For example, in the even case

$$P_+ = \int_k^1 \varphi_+(x)\,dx = -\lambda^2\pi T_+ \tag{2.8}$$

$$M_+ = \int_k^1 x\varphi_+(x)\,dx = 2\lambda^2(Q_+b_1 + R_+b_2 + T_+b_3) + \frac{2\lambda^2}{\pi}\int_k^1 \frac{x^2\,dx}{g(x)} \int_k^1 \frac{g(y)G_+'''(y)}{y^2 - x^2}\,dy$$

3. The principal term of the asymptotic form of the solution of the integral Eq.(2.1) is composed for small λ /8/ by combining the exact solution of the problem of a semi-infinite rib and the problem of one rib of finite length studied below. The function $K(u)$ from (1.1) in the Wiener-Hopf equations that occur here is approximated, as earlier, by the easily factorizable function (6.4) from /2/. In the cases $f_+(x) \equiv 1$, $f_-(x) = \operatorname{sgn} x$, for $\lambda < 1 - k$ the representation

$$\varphi_\pm(x) = \psi\left(\frac{|x| - k}{\lambda}\right)\begin{Bmatrix}1\\ \operatorname{sgn} x\end{Bmatrix} + \psi\left(\frac{1+x}{\lambda}\right) \pm \psi\left(\frac{1-x}{\lambda}\right) - \frac{0,396}{\lambda^3}\begin{Bmatrix}2\\ 0\end{Bmatrix} \tag{3.1}$$

holds where $\psi(x)$ is given by (1.4).

The forces $P_\pm$ and moments $M_\pm$ applied to the inclusions can be found for small λ by numerical integration of (3.1) in the sense of the finite part.

Certain values of $\varphi_+(x)$ computed by using the relationships (2.7) for $\lambda = 5$ (the upper row, $Q_+ = 0.006763$, $R_+ = 0.01156$, and $T_+ = -0.01378$) and by (3.1) for $\lambda = 0.5$ (the lower row) and $k = 0.1$ are given below

x	0.2	0.4	0.5	0.6	0.7	0.9
$\varphi_+(x)$	14.73	3.691	2.876	2.586	2.635	5.377
$-\varphi_+(x)$	4.299	1.321	1.154	1.153	1.318	4.284

Calculations show that the sign changes for $\varphi_-(x)$ for $\lambda = 0.5$ in the interval $k < x < 1$, indicating the possibility of separating the inclusion from the plate. This is obviously due to with the fact that the Kirchhoff-Love theory describes the state of stress rather poorly near an intense transverse load /9/. No change in the sign of $\varphi_-(x)$ is observed for $\lambda > 2$.

We find from (2.8) for $\lambda = 5$ and $k = 0.1$: $P_+ = 1.082$, $M_+ = 1.413$, and we have for $\lambda = 0.5$ and $k = 0.1$: $P_+ = 0.427$, $M_+ = 0.992$.

A numerical analysis enables us to deduce that for $\lambda \geqslant 2$ and fixed k the mutual influence of the inclusions does not change noticeably as λ increases. At the same time, it can be proved that for $\lambda < \lambda^* < 1$ the function $\varphi_+(x)$ differs from $\varphi(x)$ by less than 1%, i.e., the inclusions have practically no influence on each other ($\varphi(x)$ is the solution of the problem for one rib located in the segment $y = 0$, $k \leqslant x \leqslant 1$).

Finally, having formulas for $\varphi_\pm(x)$ and expressions for the dimensionless deflections of a plate $\omega^\pm(x, y)$ of the type (1.5), we compute the dimensionless bending moments $M_x^\pm$, $M_y^\pm$ in a certain neighbourhood of the inclusions. The deductions made about these moments in Sect.1 remain true here also, on the whole. Thus, for $k = 0.1$ and $\lambda = 5$ the function $|M_x^+|$ has a local extremum at the origin where $|M_x^+| = 9.2 \times 10^3$, and the spacing between the local extrema on this semi-axis increases significantly compared with the problem of impressing one rib for $\lambda = 5$.

The method developed above is carried over to the solution of mixed problems of plate bending in the form of an infinite strip /2/ in the case of two sections of boundary conditions interchange.

We also note that integral Eq.(2.1) with the minus sign corresponds exactly to the problem of impressing a rib into a plate lying on a Winkler foundation and having the form of a half-plane on whose boundary free support conditions are imposed. If the edge of such a plate is rigidly clamped or free of forces and moments, then the kernel of the appropriate integral equations cannot be represented successfully as was done in (2.1). Nevertheless, for sufficiently small λ these equations can be solved by successive approximations by taking the principal term of the asymptotic form (3.1) with a minus sign as the zero-th approximation.

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